

# Minimal Lagrangian surfaces in the tangent bundle of a Riemannian surface

Henri Anciaux, Brendan Guilfoyle, Pascal Romon

## Abstract

Given an oriented Riemannian surface  $(\Sigma, g)$ , its tangent bundle  $T\Sigma$  enjoys a natural pseudo-Kähler structure, that is the combination of a complex structure  $\mathbb{J}$ , a pseudo-metric  $\mathbb{G}$  with neutral signature and a symplectic structure  $\Omega$ . We give a local classification of those surfaces of  $T\Sigma$  which are both Lagrangian with respect to  $\Omega$  and minimal with respect to  $\mathbb{G}$ . We first show that if  $g$  is non-flat, the only such surfaces are affine normal bundles over geodesics. In the flat case there is, in contrast, a large set of Lagrangian minimal surfaces, which is described explicitly. As an application, we show that motions of surfaces in  $\mathbb{R}^3$  or  $\mathbb{R}_1^3$  induce Hamiltonian motions of their normal congruences, which are Lagrangian surfaces in  $T\mathbb{S}^2$  or  $T\mathbb{H}^2$  respectively. We relate the area of the congruence to a second-order functional  $\mathcal{F} = \int \sqrt{H^2 - K} dA$  on the original surface.

2000 MSC: 53A10

## Introduction

It has recently been observed (*cf.* [GK1],[GK2]) that the tangent bundle  $T\Sigma$  of an oriented Riemannian surface  $(\Sigma, g, j)$  with metric  $g$  and complex structure  $j$  enjoys a rich structure: besides the symplectic form  $\Omega$  obtained by pulling back the canonical symplectic form of  $T^*\Sigma$ , it can be endowed with a natural complex structure  $\mathbb{J}$  depending on the complex structure  $j$ ; next we may define a symmetric 2-tensor  $\mathbb{G}$  by combining  $\Omega$  and  $\mathbb{J}$  in the formula  $\mathbb{G}(\cdot, \cdot) = \Omega(\mathbb{J}\cdot, \cdot)$ . It turns out that  $\mathbb{G}$  is a pseudo-Riemannian metric on  $T\Sigma$  with signature  $(2, 2)$  and that the complex structure  $\mathbb{J}$  is parallel with respect to  $\mathbb{G}$ ; in other words we have a pseudo-Kähler structure on  $T\Sigma$ . Of

particular interest is the case of  $\Sigma$  being the two-sphere  $\mathbb{S}^2$ , since  $T\mathbb{S}^2$  can be naturally identified with the space of oriented lines of Euclidean three-space  $\mathbb{R}^3$ . Moreover, under this identification, a two-parameter family of lines in  $\mathbb{R}^3$ —thus a surface in  $T\mathbb{S}^2$ —is Lagrangian if and only if the lines are normal to some surface of  $\mathbb{R}^3$ .

Natural objects of study in Kähler geometry are minimal Lagrangian submanifolds (*cf.* [S],[SW]). In the particular case of Kähler-Einstein manifolds, it is a remarkable fact that the mean curvature vector  $\vec{H}$  of a Lagrangian submanifold  $L$  of dimension  $n$  is related by the formula  $\vec{H} = \frac{1}{n}J\nabla\beta$  to a function  $\beta$ , the *Lagrangian angle*, defined on the submanifold. A striking consequence of this formula is that a Lagrangian submanifold is minimal if and only if it has constant Lagrangian angle. From the analytical viewpoint this structure reduces the order of the corresponding PDE from 2 to 1. In the context of Calabi-Yau geometry, these submanifolds are in addition calibrated and thus minimizers, and are called *Special Lagrangian* submanifolds (*cf.* [HL]).

In this paper we give a local classification of minimal Lagrangian surfaces in  $(T\Sigma, \mathbb{J}, \mathbb{G}, \Omega)$ . It turns out that that the picture is strongly contrasted between on the one hand, the non-flat case, which is very rigid in the sense that the only non-trivial minimal Lagrangian surfaces are the normal bundles over a geodesic of  $\Sigma$  (Theorem 1), and on the other hand the flat case, where there exists a variety of minimal Lagrangian surfaces. Moreover, in Euclidean 4-space endowed with the standard pseudo-Kähler metric of signature  $(2, 2)$ , we can attach to a Lagrangian surface a kind of Lagrangian angle function, still satisfying the formula  $\vec{H} = \frac{1}{2}\mathbb{J}\nabla\beta$ , and thus whose constancy characterizes minimal Lagrangian surfaces. Finally, the underlying partial differential equation is linear and thus can be explicitly integrated (Theorem 2).

Another important class of Lagrangian surfaces are those which are critical points of the area functional restricted to Hamiltonian variations (*cf.* [SW]). The corresponding Euler equation is the vanishing of the divergence of the mean curvature vector (for the induced metric). We give some non-trivial examples of Hamiltonian stationary Lagrangian surfaces in  $(T\Sigma, \mathbb{J}, \mathbb{G}, \Omega)$ . In the special case of  $T\mathbb{S}^2$ , we already know that these Hamiltonian stationary Lagrangian surfaces are normal congruences to some surfaces of  $\mathbb{R}^3$ . We get as a corollary that a developable surface of  $\mathbb{R}^3$  is a critical point of the second-order functional  $\mathcal{F}(S) := \int_S \sqrt{H^2 - K} dA$ . Things work exactly in the same

way with  $T\mathbb{H}^2$ , which can be identified with the set of oriented time-like lines of the Minkowski three-space  $\mathbb{R}^{2,1}$ , and whose Lagrangian surfaces are normal congruences to space-like surfaces. We thus get that a developable space-like surface of  $\mathbb{R}^{2,1}$  is a critical point of the functional equivalent to  $\mathcal{F}$  in  $\mathbb{R}^{2,1}$ . In a forthcoming paper we shall study more deeply the Hamiltonian stationary Lagrangian surfaces of  $(T\Sigma, \mathbb{J}, \mathbb{G}, \Omega)$ .

The paper is organised as follows: in Section 1 we give some preliminary results and the precise statements of the two main theorems. Section 2 is devoted to the proof of Theorem 1. The last two sections deal with special cases: in Section 3 consider the Euclidean case and prove Theorem 2; in Section 4 we take a closer look to the special cases  $T\mathbb{S}^2$  and  $T\mathbb{H}^2$ .

Finally the Authors wish to mention recent related results in the special Lagrangian case obtained independently by Dong [Dg].

# 1 Preliminaries and statements of results

## 1.1 The structures of $T\Sigma$

In the following we consider an oriented Riemannian surface  $(\Sigma, g)$  and denote by  $j$  the canonical complex structure associated to it. We denote by  $\pi$  the canonical projection of the tangent bundle  $T\Sigma$  onto its base  $\pi : T\Sigma \rightarrow \Sigma$ . The two-dimensional subbundle  $\text{Ker}(d\pi)$  of  $TT\Sigma$  (it is thus a bundle over  $T\Sigma$ ) will be called *the vertical bundle* and denoted by  $V\Sigma$ .

We observe that we have not used the metric  $g$  so far. The next step consists of using the Levi-Civita connection  $\nabla$  of  $g$  to define the *horizontal bundle*  $H\Sigma$  as follows: let  $X$  be a tangent vector to  $T\Sigma$  at some point  $(p_0, V_0)$ . This implies that there exists a curve  $\alpha(s) = (p(s), V(s))$  such that  $(\gamma(0), V(0)) = (p_0, V_0)$  and  $\alpha'(0) = X$ . If  $X \notin V\Sigma$  (which implies  $p'(0) \neq 0$ ), we define the connection map (cf. [Ko],[Do])  $K : TT\Sigma \rightarrow T\Sigma$  by  $KX = \nabla_{p'(0)} V(0)$ , which does not depend on the curve  $\alpha$ . If  $X$  is vertical, we may assume that the curve  $\alpha$  stays in a fiber so that  $V(s)$  is a curve in a vector space. We then define  $KX$  to be simply  $V'(0)$ . The horizontal bundle is then  $\text{Ker}(K)$  and we have a direct sum

$$\begin{aligned} TT\Sigma = H\Sigma \oplus V\Sigma &\simeq T\Sigma \oplus T\Sigma \\ X &\simeq (PX, KX) \end{aligned} \tag{1}$$

Here and in the following,  $P$  is a shorthand notation for  $d\pi$ . We refer to [Ko] and [Do] for a more complete description of the horizontal and vertical bundles.

We shall use again the metric  $g$  in order to pull back the canonical symplectic form of  $T^*\Sigma$  to a symplectic form  $\Omega$  in  $T\Sigma$ , which admits a nice expression in terms of the direct decomposition of  $TT\Sigma$ :

**Lemma 1.1** *Let  $X$  and  $Y$  be two tangent vectors to  $T\Sigma$ ; we have*

$$\Omega(X, Y) := g(KX, PY) - g(PX, KY).$$

A proof of this lemma can be found in [La], p. 89.

We recall then the classical

**Definition 1.2** *A surface  $L$  of  $(T\Sigma, \Omega)$  is said to be Lagrangian if the restriction of  $\Omega$  vanishes on it.*

Next we define an almost complex structure  $\mathbb{J}$  by  $\mathbb{J} = j \oplus j$ , using the direct sum (1) and the pseudo-metric  $\mathbb{G}$  by the formula  $\mathbb{G}(\cdot, \cdot) = \Omega(\mathbb{J}\cdot, \cdot)$ . In [GK1] it has been proved that  $\mathbb{G}$  is a pseudo-Riemannian metric with signature  $(2, 2)$ . Proposition 1.4 below shows that  $\mathbb{J}$  is actually a complex structure.

## 1.2 Statements of the main theorems

The projection map  $\pi : T\Sigma \rightarrow \Sigma$  plays a crucial role in the local classification of minimal Lagrangian surfaces of  $(T\Sigma, \mathbb{J}, \mathbb{G}, \Omega)$ . If  $L$  is some surface of  $T\Sigma$  (not necessarily Lagrangian), then the rank of the restriction to  $L$  of the projection  $\pi$  can be 0, 1 or 2 and is locally constant. The case of rank 0 corresponds to the trivial case of  $L$  being a piece of a vertical fibre.

A simple example of Lagrangian surface of rank 1 is the *normal bundle* over some curve  $\gamma$  of  $\Sigma$ , i.e. the set of its normal lines to the curve  $\gamma$ . More precisely, denoting by  $\vec{n}(s)$  a unit normal vector to the curve at the point  $\gamma(s)$ , the normal bundle of  $\gamma$  is the image of the immersion  $X(s, t) = (\gamma(s), t\vec{n}(s))$ . One can slightly generalize the construction by considering affine lines, i.e. adding a translation term to the second factor of the immersion:  $X(s, t) = (\gamma(s), a(s)\vec{t} + t\vec{n}(s))$ , where  $\vec{t}$  denotes the unit tangent vector to  $\gamma(s)$  and  $a(s)$  is some real-valued map. We shall call the image of such an immersion, which

is still Lagrangian, an *affine normal bundle over  $\gamma$* . Affine normal bundles and their higher dimensional equivalents have been introduced in the flat case in [HL], where they were called *degenerate projections*.

As the metric  $\mathbb{G}$  is neutral, the induced metric on a surface of  $(T\Sigma, \mathbb{J}, \mathbb{G}, \Omega)$  may be degenerate. It is for example the case of a vertical fibre and of the zero section  $L_0 := \{(p, 0), p \in \Sigma\} \subset T\Sigma$ . Such surfaces are called *null*.

The first main result of this article characterizes rank one minimal Lagrangian surfaces and shows that, beyond the null surfaces, there is no rank two Lagrangian minimal surface if  $(\Sigma, g)$  is non-flat:

**Theorem 1** *Let  $L$  be a smooth, non-null minimal Lagrangian surface of  $(T\Sigma, \mathbb{J}, \mathbb{G}, \Omega)$ . Then*

- (i) *either  $L$  is the normal bundle over a geodesic on  $(\Sigma, g)$ , or*
- (ii)  *$(\Sigma, g)$  is flat.*

The Lagrangian assumption in the theorem above is crucial: the existence of families of (non-Lagrangian) minimal surfaces in  $(T\Sigma, \mathbb{J}, \mathbb{G}, \Omega)$  has been proved in [GK2]. When the surface  $\Sigma$  is flat, the situation appears to be richer, in the sense that there exist many rank two minimal Lagrangian surfaces. As our classification is local, there is no loss of generality to restrict ourselves to the Euclidean plane.

**Theorem 2** *In the case where  $(\Sigma, g, j)$  is the Euclidean plane  $\mathbb{R}^2$  endowed with its canonical inner product  $\langle \cdot, \cdot \rangle$ , the metric  $\mathbb{G}$  on  $T\mathbb{R}^2 \simeq \mathbb{R}^4$  is the flat pseudo-metric of signature  $(2, 2)$ . Moreover, if  $L$  is a rank two minimal Lagrangian surface of  $(T\mathbb{R}^2, \mathbb{J}, \mathbb{G}, \Omega)$ , then it is parametrized by  $X(p) = (p, \nabla u(p))$ , where the real map  $u$  takes the following form*

$$u(p) = f_1(\langle p, V \rangle) + f_2(\langle p, jV \rangle),$$

*where  $V$  is some constant unit vector of  $\mathbb{R}^2$  and  $f_1$  and  $f_2$  are two non-constant functions of the real variable of class  $C^2$ .*

In Section 3 we shall give, along with the proof of the theorem, a geometric interpretation of the vector  $V$ .

### 1.3 Some preliminary results

We start with a result from [Ko] which will be useful:

**Lemma 1.3** [Ko] *Given a vector field  $X$  on  $(\Sigma, g)$ , there exists exactly one vector field  $X^h$  and one vector field  $X^v$  on  $T\Sigma$  <sup>(1)</sup> such that  $(PX^h, KX^h) = (X, 0)$  and  $(PX^v, KX^v) = (0, X)$ . Moreover, given two vector fields  $X$  and  $Y$  on  $(\Sigma, g)$ , we have, at the point  $(p, V)$ :*

$$[X^v, Y^v] = 0$$

$$[X^h, Y^v] \simeq (0, \nabla_X Y)$$

$$[X^h, Y^h] \simeq ([X, Y], -R(X, Y)V),$$

where  $R$  denotes the curvature of  $g$  and we use the direct sum notation (1).

We say that a vector field  $X$  on  $T\Sigma$  is *projectable* if it is constant on the fibres. According to the lemma above, it is equivalent to the fact that there exists two vector fields  $X_1$  and  $X_2$  on  $\Sigma$  such that  $X = (X_1)^h + (X_2)^v$ .

We can now prove

**Proposition 1.4** *The almost complex structure  $\mathbb{J}$  is complex.*

*Proof.* We compute the Nijenhuis tensor

$$N(X, Y) = [X, Y] + \mathbb{J}[\mathbb{J}X, Y] + \mathbb{J}[X, \mathbb{J}Y] - [\mathbb{J}X, \mathbb{J}Y].$$

Since  $N(X, Y)$  depends pointwise on the tangent vectors we may assume for computational purposes that  $X$  and  $Y$  are projectable, and use lemma 1.3 together with the definition  $\mathbb{J} = j \oplus j$ . By linearity and skew-symmetry it suffices to prove that  $N(X, Y) = 0$  in three distinct cases:

1. vertical fields

$$N(X^v, Y^v) = 0$$

---

<sup>1</sup>The Reader should be aware that the notation for  $X^v, X^h$  in Lemma 1.3 follows [Ko] and corresponds to particular lifts of a field on  $M$ , whereas [La], in the proof of Lemma 1.1, uses the same notation to denote *projections* of a vector field.

2. horizontal fields

$$\begin{aligned}
N(X^h, Y^h) &= -\left(0, R(X, Y)V + jR(jX, Y)V \right. \\
&\quad \left. + jR(X, jY)V - R(jX, jY)V \right) \\
&= -(0, jR(jX, Y)V + jR(X, jY)V) = (0, 0)
\end{aligned}$$

3. mixed fields

$$\begin{aligned}
N(X^h, Y^v) &= (0, \nabla_X Y + j\nabla_{jX} Y + j\nabla_X(jY) - \nabla_{jX}(jY)) \\
&= (0, \nabla_X Y + j\nabla_{jX} Y - \nabla_X Y - j\nabla_{jX} Y) = (0, 0)
\end{aligned}$$

where we have used the properties of Kähler manifolds:  $\nabla j = 0$ ,  $R(jX, jY) = R(X, Y)$ . ■

**Corollary 1.5** *The triple  $(\mathbb{G}, \mathbb{J}, \Omega)$  defines a pseudo-Kähler structure on  $T\Sigma$ . In particular  $\mathbb{J}$  is parallel for the Levi-Civita connection.*

The following lemma describes the Levi-Civita connection  $D$  of  $\mathbb{G}$  in terms of the direct decomposition of  $TT\Sigma$ .

**Lemma 1.6** *Let  $X$  and  $Y$  two vector fields and assume that  $Y$  is projectable, then at the point  $(p, V)$  we have*

$$D_X Y = \begin{pmatrix} \nabla_{PX} PY \\ \nabla_{PX} KY - \frac{1}{2} \left( R(PX, PY)V - jR(V, jPX)PY - jR(V, jPY)PX \right) \end{pmatrix}$$

where we have used column vector notation to indicate the components in the direct sum (1).

*Proof.* We use Lemma 1.3 together with the Koszul formula:

$$\begin{aligned}
2\mathbb{G}(D_X Y, Z) &= X\mathbb{G}(Y, Z) + Y\mathbb{G}(X, Z) - Z\mathbb{G}(X, Y) + \mathbb{G}([X, Y], Z) \\
&\quad - \mathbb{G}([X, Z], Y) - \mathbb{G}([Y, Z], X)
\end{aligned}$$

where  $X, Y$  and  $Z$  are three vector fields on  $T\Sigma$ . From the fact that  $[X^v, Y^v]$  and  $\mathbb{G}(X^v, Y^v)$  vanish we have:

$$\begin{aligned}
2\mathbb{G}(D_{X^v} Y^v, Z^v) &= X^v\mathbb{G}(Y^v, Z^v) + Y^v\mathbb{G}(X^v, Z^v) - Z^v\mathbb{G}(X^v, Y^v) \\
&\quad + \mathbb{G}([X^v, Y^v], Z^v) - \mathbb{G}([X^v, Z^v], Y^v) - \mathbb{G}([Y^v, Z^v], X^v) = 0.
\end{aligned}$$

Moreover, taking into account that  $\mathbb{G}(Y^v, Z^h)$  and similar quantities are constant on the fibres, we obtain

$$\begin{aligned} 2\mathbb{G}(D_{X^v}Y^v, Z^h) &= X^v\mathbb{G}(Y^v, Z^h) + Y^v\mathbb{G}(X^v, Z^h) - Z^h\mathbb{G}(X^v, Y^v) \\ &\quad + \mathbb{G}([X^v, Y^v], Z^h) - \mathbb{G}([X^v, Z^h], Y^v) - \mathbb{G}([Y^v, Z^h], X^v) \\ &= -\mathbb{G}(-(\nabla_Z X)^v, Y^v) - \mathbb{G}(-(\nabla_Z Y)^v, X^v) = 0. \end{aligned}$$

From these last two equations we deduce that  $D_{X^v}Y^v$  vanishes. Analogous computations show that  $D_{X^v}Y^h$  vanishes as well. From Lemma 1.3 and the formula  $[X, Y] = D_X Y - D_Y X$ , we deduce that  $D_{X^h}Y^v \simeq (0, \nabla_X Y)$ .

Finally, introducing  $jW = -jR(X, Y)V - R(V, jY)X - R(V, jX)Y$ , we compute that

$$\mathbb{G}(D_{X^h}Y^h, Z^h) = \frac{1}{2}g(jW, Z)$$

and

$$\mathbb{G}(D_{X^h}Y^h, Z^v) = g(jZ, \nabla_X Y),$$

from which we deduce that

$$\begin{aligned} D_{X^h}Y^h &= (\nabla_X Y, \frac{1}{2}W) \\ &= \left( \nabla_X Y, \frac{1}{2}(-R(X, Y)V + jR(V, jX)Y + jR(V, jY)X) \right). \end{aligned}$$

The conclusion of the proof follows easily. ■

The fact that  $\mathbb{J}$  is parallel with respect to  $D$  implies the following useful result about the extrinsic geometry of Lagrangian surfaces; this fact is known to hold in a positive Kähler manifold, *cf.* [Ch].

**Lemma 1.7** *Let  $L$  be a Lagrangian surface of  $T\Sigma$  and  $X, Y$  and  $Z$  three vector fields tangent to  $L$ . Then*

$$h(X, Y, Z) := \Omega(X, D_Y Z) = \mathbb{G}(\mathbb{J}X, D_Y Z)$$

*defines a tri-symmetric tensor called the tensor of extrinsic curvature.*

*Proof.* Let  $\text{II}$  denote the second fundamental form of the immersion:

$$h(X, Y, Z) = \mathbb{G}(\mathbb{J}X, D_Y Z) = \mathbb{G}(\mathbb{J}X, \text{II}(Y, Z)),$$



which proves the tensorial nature of  $h$  as well as the symmetry with respect to its last two variables.

$$\begin{aligned} h(X, Y, Z) &= Y\mathbb{G}(\mathbb{J}X, Z) - \mathbb{G}(D_Y\mathbb{J}X, Z) = -\mathbb{G}(\mathbb{J}D_YX, Z) = \mathbb{G}(\mathbb{J}Z, D_YX) \\ &= h(Z, Y, X) \end{aligned}$$

using the Lagrangian hypothesis on  $L$ . ■

## 2 Proof of Theorem 1

The proof of Theorem 1 will result from Propositions 2.1 and 2.2, dealing with Lagrangian surfaces of rank one and two, respectively.

### 2.1 Rank one Lagrangian surfaces

**Proposition 2.1** *A rank one Lagrangian surface  $L$  of  $(T\Sigma, \mathbb{J}, \mathbb{G}, \Omega)$  is an affine normal bundle over a curve  $\gamma$  of  $\Sigma$ . It is moreover  $H$ -minimal and the induced metric on  $L$  is flat. Finally,  $L$  is minimal if and only if the base curve  $\gamma$  is a geodesic of  $(\Sigma, g)$ .*

*Proof.* A surface  $L$  of  $T\Sigma$  with rank 1 projection may be parametrized locally by

$$\begin{aligned} X : U &\rightarrow T\Sigma \\ (s, t) &\mapsto (\gamma(s), V(s, t)), \end{aligned}$$

where  $\gamma(s)$  is a regular curve in  $\Sigma$  and  $V(s, t)$  some tangent vector to  $\Sigma$  at the point  $\gamma(s)$ . Without loss of generality, we may assume that  $\gamma$  is parametrized by arclength, so that  $\{\gamma'(s), j\gamma'(s)\}$  is an orthonormal frame of  $T\Sigma$  along the curve  $\gamma$ . Writing  $V = a\gamma' + bj\gamma'$  and using the Frénet equation  $\nabla_{\gamma'}\gamma' = k j\gamma'$ , where  $k$  denotes the curvature of  $\gamma$ , we compute the first derivatives of the immersion (here and in the following, a letter in subscript denotes partial differentiation with respect to the corresponding variable). Using the direct sum notation:

$$\begin{aligned} X_s &= (PX_s, KX_s) = (\gamma', \nabla_{\gamma'}\gamma') = (\gamma', (a_s - kb)\gamma' + (b_s + ka)j\gamma') \\ X_t &= (0, a_t\gamma' + b_tj\gamma'). \end{aligned}$$

If the immersion is Lagrangian, the following must vanish:

$$\Omega(X_s, X_t) = -g(\gamma', a_t \gamma' + b_t j \gamma') = -a_t.$$

It follows that  $a$  must be a function of  $s$ . Then we see that for fixed  $s$ , the map  $t \mapsto (\gamma(s), a(s)\gamma'(s) + b(s, t)j\gamma'(s))$  parametrizes a line segment ruled by  $j\gamma'(s)$  in  $T_{\gamma(s)}\Sigma$  which may be reparametrized by

$$t \mapsto (\gamma(s), a(s)\gamma'(s) + tj\gamma'(s))$$

which we assume henceforth. We have thus proved the first part of Proposition 2.1.

We compute easily that  $X_s = (\gamma', (a' - kt)\gamma' + akj\gamma')$  and  $X_t = (0, j\gamma')$ . We also observe that the vector field (defined along the surface)  $X_t$  depends only on the variable  $s$ , thus it can be extended to a global vector field which is projectable. It follows that we can use Lemma 1.6 in order to compute:

$$D_{X_s}X_t = (0, \nabla_{\gamma'}j\gamma') = (0, -k\gamma'), \quad D_{X_t}X_t = (0, 0).$$

In view of Lemma 1.7, the symmetric tensor  $h(X, Y, Z)$  has four independent components. We calculate:

$$h_{112} = \Omega(X_s, D_{X_s}X_t) = \Omega((\gamma', (a' - kt)\gamma' + akj\gamma'), (0, -k\gamma')) = k$$

$$h_{122} = \Omega(X_s, D_{X_t}X_t) = 0 \quad h_{222} = \Omega(X_t, D_{X_t}X_t) = 0.$$

(As will become clear in a moment, we do not need the expression of  $h_{111}$ .)

It remains to compute the induced metric, which is given in the coordinates  $(s, t)$  by

$$\begin{pmatrix} -2ak & -1 \\ -1 & 0 \end{pmatrix}.$$

We are now in a position to get the expression of the mean curvature vector:

$$\mathbb{G}(2\vec{H}, \mathbb{J}X_s) = \frac{h_{111}G + h_{122}E - 2h_{112}F}{EG - F^2} = -2k$$

and

$$\mathbb{G}(2\vec{H}, \mathbb{J}X_t) = \frac{h_{112}G + h_{222}E - 2h_{122}F}{EG - F^2} = 0.$$

It follows that

$$\vec{H} = k\mathbb{J}X_t = (0, kj\gamma') = (0, \gamma''(s)),$$

so that  $L$  is minimal if and only if  $k$  vanishes, namely  $\gamma$  is a geodesic. Moreover, the determinant of the induced metric being  $-1$ , we have the following formula:

$$\operatorname{div} \mathbb{J} \vec{H} = \operatorname{div}(-k \partial_t) = 0$$

and hence  $L$  is always Hamiltonian stationary. Finally, denoting by  $\bar{\nabla}$  and  $\bar{R}$  the Levi-Civita connection and the curvature of the induced metric, an easy computation shows that  $\bar{\nabla}_{\partial_s} \partial_t$  and  $\bar{\nabla}_{\partial_t} \partial_t$  vanish, so that

$$\bar{R}(\partial_t, \partial_s) \partial_t = \bar{\nabla}_{\partial_t} \bar{\nabla}_{\partial_s} \partial_t - \bar{\nabla}_{\partial_s} \bar{\nabla}_{\partial_t} \partial_t = 0,$$

which implies the flatness of  $L$ . ■

## 2.2 Rank two Lagrangian surfaces

**Proposition 2.2** *A rank two Lagrangian surface  $L$  of  $(T\Sigma, \mathbb{J}, \mathbb{G}, \Omega)$  is the graph of the gradient of a real map  $u$  on  $(\Sigma, g)$ :*

$$L := \{(p, \nabla u(p)), p \in \Sigma\} \subset T\Sigma.$$

*Moreover, if  $L$  is minimal then  $g$  is flat.*

*Proof.* A rank 2 surface is nothing but the graph of a vector field  $V(p)$  of  $\Sigma$  and thus is the image of the immersion  $X(p) = (p, V(p))$ . Let  $(s, t)$  be conformal local coordinates on  $(\Sigma, g)$  such that  $j\partial_s = \partial_t$  and  $j\partial_t = -\partial_s$ . We denote by  $r(s, t)$  the logarithmic conformal factor, so that the metric takes the following form:  $g(s, t) = e^{2r}(ds^2 + dt^2)$ . A standard computation shows that

$$\begin{aligned} \nabla_{\partial_s} \partial_s &= r_s \partial_s - r_t \partial_t \\ \nabla_{\partial_t} \partial_s &= \nabla_{\partial_s} \partial_t = r_t \partial_s + r_s \partial_t \\ \nabla_{\partial_t} \partial_t &= -r_s \partial_s + r_t \partial_t. \end{aligned}$$

The following relations between the curvature tensor  $R$ , the Gauss curvature  $K$  and the conformal factor  $r$  of  $(\Sigma, g)$  will be useful later:

$$K = e^{-4r} g(R(\partial_s, \partial_t) \partial_s, \partial_t) = -e^{-2r} \Delta r.$$

Writing  $V(s, t) = P(s, t) \partial_s + Q(s, t) \partial_t$ , the first derivatives of the immersion are:

$$X_s = (\partial_s, (P_s + Pr_s + Qr_t) \partial_s + (Q_s - Pr_t + Qr_s) \partial_t),$$

$$X_t = (\partial_t, (P_t + Pr_t - Qr_s)\partial_s + (Q_t + Pr_s + Qr_t)\partial_t),$$

so that

$$\begin{aligned}\Omega(X_s, X_t) &= g((Q_s - Pr_t + Qr_s)\partial_t, \partial_t) - g(\partial_s, (P_t + Pr_t - Qr_s)\partial_s) \\ &= e^{2r}(Q_s + 2Qr_s - P_t - 2Pr_t).\end{aligned}$$

Thus the Lagrangian condition is equivalent to  $(Pe^{2r})_t = (Qe^{2r})_s$ , so that there exists locally a real map  $u$  on  $\Sigma$  such that  $Pe^{2r} = u_s$  and  $Qe^{2r} = u_t$ ; in other words, the vector field  $V$  is the gradient of  $u$ , and we have the first part of Proposition 2.2.

Next a parametrization of  $L$  is

$$\begin{aligned}X : \Sigma &\rightarrow T\Sigma \\ (s, t) &\mapsto (p(s, t), e^{-2r}(u_s\partial_s + u_t\partial_t)),\end{aligned}$$

and we compute

$$\begin{aligned}X_s &= (\partial_s, \nabla_{\partial_s}\nabla u) \\ &= (\partial_s, e^{-2r}((u_{ss} - 2r_s u_s)\partial_s + u_s \nabla_{\partial_s}\partial_s + (u_{st} - 2r_s u_t)\partial_t + u_t \nabla_{\partial_s}\partial_t)) \\ &= (\partial_s, e^{-2r}(u_{ss} - r_s u_s + r_t u_t)\partial_s + e^{-2r}(u_{st} - r_s u_t - r_t u_s)\partial_t)\end{aligned}$$

Analogously

$$X_t = (\partial_t, e^{-2r}(u_{st} - r_s u_t - r_t u_s)\partial_s + e^{-2r}(u_{tt} - r_t u_t + r_s u_s)\partial_t).$$

Denoting for simplicity

$$X_s := (\partial_s, a\partial_s + b\partial_t) \quad X_t := (\partial_t, b\partial_s + c\partial_t),$$

the induced metric is given by

$$E = \mathbb{G}(X_s, X_s) = \Omega(\mathbb{J}X_s, X_s) = g(j(a\partial_s + b\partial_t), \partial_s) - g(j\partial_s, a\partial_s + b\partial_t) = -2be^{2r},$$

$$F = \Omega(\mathbb{J}X_s, X_t) = g(j(a\partial_s + b\partial_t), \partial_t) - g(j\partial_s, b\partial_s + c\partial_t) = (a - c)e^{2r},$$

$$G = \Omega(\mathbb{J}X_t, X_t) = g(j(b\partial_s + c\partial_t), \partial_t) - g(j\partial_t, b\partial_s + c\partial_t) = 2be^{2r}.$$

Moreover, the vector fields  $X_s$  and  $X_t$  admit extensions on  $T\Sigma$  which are projectable, to that we can use Lemma 1.6, to get

$$D_{X_s}X_s = (r_s\partial_s - r_t\partial_t, (a_s + ar_s + br_t)\partial_s + (b_s - ar_t + br_s)\partial_t + u_s e^{-2r}jR(\partial_s, \partial_t)\partial_s)$$

$$D_{X_s}X_t = \left( r_t\partial_s + r_s\partial_t, (b_s + br_s + cr_t)\partial_s + (c_s - br_t + cr_s)\partial_t \right) \\ + \frac{1}{2} \left( 0, -R(\partial_s, \partial_t)\nabla u + u_se^{-2r}jR(\partial_s, \partial_t)\partial_t + u_te^{-2r}jR(\partial_s, \partial_t)\partial_s \right),$$

$$D_{X_t}X_t = \left( -r_s\partial_s + r_t\partial_t, (b_t + br_t - cr_s)\partial_s + (c_t + br_s + c_t)\partial_t - u_te^{-2r}jR(\partial_t, \partial_s)\partial_t \right).$$

This allows us to calculate the following components of the tensor  $h$ :

$$\begin{aligned} h_{111} &= h(X_s, X_s, X_s) = \Omega(X_s, D_{X_s}X_s) \\ &= g(a\partial_s + b\partial_t, r_s\partial_s - r_t\partial_t) \\ &\quad - g(\partial_s, (a_s + ar_s + br_t)\partial_s) - u_se^{-2r}g(\partial_s, jR(\partial_s, \partial_t)\partial_s) \\ &= e^{2r}(ar_s - br_t - (a_s + ar_s + br_t)) + u_se^{-2r}g(\partial_t, R(\partial_s, \partial_t)\partial_s) \\ &= e^{2r}(-a_s - 2br_t + u_sK). \end{aligned}$$

and similarly<sup>2</sup> with the other coefficients of  $h$ . Consequently

$$\begin{aligned} \mathbb{G}(2\vec{H}, \mathbb{J}X_s) &= \frac{h_{111}G + h_{122}E - 2h_{112}F}{EG - F^2} = \frac{2b(h_{111} - h_{122}) - 2(a - c)h_{112}}{-e^{2r}(4b^2 + (a - c)^2)} \\ &= \frac{2b((a - c)_s + 4br_t) + 2(a - c)(-b_s + (a - c)r_t)}{4b^2 + (a - c)^2} \\ &= \frac{(a - c)_s(2b) - (a - c)2b_s}{(2b)^2 + (a - c)^2} + 2r_t = (\arg(2b + i(a - c)))_s + 2r_t. \end{aligned}$$

A similar computation yields

$$\mathbb{G}(2\vec{H}, \mathbb{J}X_t) = (\arg(a - c) + 2ib)_t - 2r_s,$$

and hence the vanishing of  $\vec{H}$  implies

$$\begin{aligned} (\arg(c - a + 2ib))_s - 2r_t &= 0 \\ (\arg(c - a + 2ib))_t + 2r_s &= 0. \end{aligned}$$

Differentiating the first equation with respect to the variable  $t$ , and the second equation with respect to the variable  $s$  yields  $\Delta r = 0$ , which implies that  $\Sigma$  has vanishing curvature and concludes the proof of Proposition 2.2.  $\blacksquare$

---

<sup>2</sup> Note that coefficients  $h_{112}$  and  $h_{122}$  can be computed by two different methods, yielding two seemingly different expressions.

### 3 Minimal Lagrangian surfaces in $T\mathbb{R}^2$

We now consider the Euclidean plane  $(\mathbb{R}^2, \langle \cdot, \cdot \rangle)$  with coordinates  $(x_1, x_2)$ ; the metric is  $dx_1^2 + dx_2^2$  and the complex structure is  $j(x_1, x_2) = (-x_2, x_1)$ . On  $T\mathbb{R}^2$  we define the coordinates  $(x_1, x_2, y_1, y_2)$ , in which the canonical symplectic structure writes

$$\Omega = dy_1 \wedge dx_1 + dy_2 \wedge dx_2.$$

The complex structure is  $\mathbb{J} := j \oplus j$  and the metric  $\mathbb{G}$  by

$$\mathbb{G} = dx_2 dy_1 - dx_1 dy_2.$$

The complex structure  $\mathbb{J}$  induces an identification of  $T\mathbb{R}^2 \simeq \mathbb{R}^{2,2}$  with  $\mathbb{C}^2$ , given by  $(x_1, x_2, y_1, y_2) \simeq (w_1 := x_1 + ix_2, w_2 := y_1 + iy_2)$ . The pseudo-Hermitian metric takes the form:

$$H(\cdot, \cdot) = \mathbb{G}(\cdot, \cdot) + i\Omega(\cdot, \cdot) = \mathbb{G}(\cdot, \cdot) + i\mathbb{G}(\cdot, \mathbb{J}\cdot).$$

However we can consider on  $\mathbb{C}^2 \simeq T\mathbb{R}^2$  the canonical Riemannian and symplectic structures, and define classically the Lagrangian angle: if  $e_1 \wedge e_2$  denotes a Lagrangian plane, its Lagrangian angle  $\beta$  is the argument of  $dw_1 \wedge dw_2(e_1 \wedge e_2) = \det_{\mathbb{C}}(e_1, e_2)$ . If  $L$  is some Lagrangian surface, the Lagrangian angle function  $\beta$  is defined on  $L$  by  $\beta(p) = \beta(T_p L)$ . The Reader should note that  $\beta$  bears a priori no relation to the pseudo-Kählerian structure defined on  $T\mathbb{R}^2$ . Nevertheless one obtains the following surprising

**Proposition 3.1** *The relation*

$$\vec{H} = \frac{1}{2} \mathbb{J} D\beta$$

*still holds for Lagrangian surfaces of  $\mathbb{R}^{2,2}$ , where  $D\beta$  denotes the gradient of  $\beta$  in the induced (pseudo-)metric.*

*In particular, a Lagrangian surface is minimal if and only if its Lagrangian angle is (locally) constant.*

*Proof.* Let  $(e_1, e_2)$  a frame along  $L$  such that  $\mathbb{G}(e_1, e_1) = -1, \mathbb{G}(e_2, e_2) = 1$  and  $\mathbb{G}(e_1, e_2) = 0$ . The Lagrangian assumption implies  $H(e_1, e_1) = -1$ ,

$H(e_2, e_2) = 1$  and  $H(e_1, e_2) = 0$ . Thus, given a vector  $\vec{V}$  of  $\mathbb{R}^4$ , the following formula holds:

$$\vec{V} = -H(\vec{V}, e_1)e_1 + H(\vec{V}, e_2)e_2.$$

We differentiate the relation  $e^{i\beta(p)} = \det_{\mathbb{C}}(e_1(p), e_2(p))$  with respect to  $e_1$ , which yields, using the fact that  $dw_1 \wedge dw_2$  is parallel with respect to the Levi-Civita connection induced by  $\mathbb{G}$ :

$$\begin{aligned} ie_1(\beta)e^{i\beta} &= \det_{\mathbb{C}}(D_{e_1}e_1, e_2) + \det_{\mathbb{C}}(e_1, D_{e_1}e_2) \\ &= -H(D_{e_1}e_1, e_1)\det_{\mathbb{C}}(e_1, e_2) + H(D_{e_1}e_2, e_2)\det_{\mathbb{C}}(e_1, e_2) \\ &= e^{i\beta} [(-\mathbb{G}(D_{e_1}e_1, e_1) + \mathbb{G}(D_{e_1}e_2, e_2)) \\ &\quad + i(-\mathbb{G}(D_{e_1}e_1, \mathbb{J}e_1) + \mathbb{G}(D_{e_1}e_2, \mathbb{J}e_2))]. \end{aligned}$$

Thus  $e_1(\beta) = -h(e_1, e_1, e_1) + h(e_1, e_2, e_2) = \mathbb{G}(2\vec{H}, \mathbb{J}e_1)$ , proving that

$$\mathbb{G}(\mathbb{J}D\beta, \mathbb{J}e_1) = \mathbb{G}(2\vec{H}, \mathbb{J}e_1).$$

Analogously we prove that  $\mathbb{G}(\mathbb{J}D\beta, \mathbb{J}e_2) = \mathbb{G}(2\vec{H}, \mathbb{J}e_2)$  and the proof is complete.  $\blacksquare$

**Remark 3.2** *The surprising fact that one uses the same definition for the Lagrangian angle though the underlying (pseudo-)Kähler is quite different can be explained by looking at the isometries for that structure. Indeed the group is  $(\mathbb{G}, \mathbb{J}, \Omega)$ -preserving matrices is none other than  $U(1) \times SL(2, \mathbb{R})$ , written in complex notations as  $2 \times 2$  matrices. (While the corresponding group in flat  $\mathbb{C}^2$  is  $U(2) = U(1) \times SU(2)$ .) So that in both cases the Lagrangian angle measures the  $U(1)$  factor.*

We are now in position to determine locally the minimal Lagrangian surfaces of  $\mathbb{R}^{2,2}$ :

**Proposition 3.3** *Let  $L$  be a rank two Lagrangian surface of  $\mathbb{R}^{2,2}$ , i.e. it is parametrized by  $X(p) = (p, \nabla u(p))$ , where  $u$  is a  $C^2$  map defined on an open subset of  $(\mathbb{R}^2, \langle \cdot, \cdot \rangle)$ . Then  $L$  has constant Lagrangian angle  $\beta_0$  if and only if it takes the following form*

$$u(p) = f_1(\langle p, e^{i(\beta_0/2 + \pi/4)} \rangle) + f_2(\langle p, ie^{i(\beta_0/2 + \pi/4)} \rangle),$$

where  $f_1$  and  $f_2$  are two non-constant functions of the real variable of class  $C^2$ .

*Proof.* We first compute the first derivatives of the immersion  $X$ , writing  $p = (s, t)$ :

$$X_s = (1, 0, u_{ss}, u_{st}) \simeq (1, u_{ss} + iu_{st}) \quad X_t = (0, 1, u_{st}, u_{tt}) \simeq (i, u_{st} + iu_{tt}),$$

so the Lagrangian angle map is given by:

$$\beta(s, t) = \arctan\left(\frac{u_{tt} - u_{ss}}{2u_{st}}\right)$$

and the constant Lagrangian angle condition translates into the linear PDE

$$\cos \beta_0(u_{tt} - u_{ss}) - 2 \sin \beta_0 u_{st} = 0,$$

In order to solve this, we introduce the linear change of variables defined by

$$\begin{pmatrix} \sigma \\ \tau \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix},$$

where  $\theta$  is some fixed constant, so that

$$\begin{aligned} u_{tt} &= \sin^2 \theta u_{\sigma\sigma} + \cos^2 \theta u_{\tau\tau} + 2 \cos \theta \sin \theta u_{\sigma\tau}, \\ u_{ss} &= \cos^2 \theta u_{\sigma\sigma} + \sin^2 \theta u_{\tau\tau} - 2 \cos \theta \sin \theta u_{\sigma\tau}, \\ u_{st} &= (\cos^2 \theta - \sin^2 \theta) u_{\sigma\tau} + \cos \theta \sin \theta (u_{\sigma\sigma} - u_{\tau\tau}), \end{aligned}$$

and thus

$$\begin{aligned} \cos \beta_0(u_{tt} - u_{ss}) - 2 \sin \beta_0 u_{st} &= (\cos \beta_0(\sin^2 \theta - \cos^2 \theta) - 2 \sin \beta_0 \cos \theta \sin \theta) u_{\sigma\sigma} \\ &\quad + (\cos \beta_0(\cos^2 \theta - \sin^2 \theta) + 2 \sin \beta_0 \cos \theta \sin \theta) u_{\tau\tau} \\ &\quad + (4 \cos \beta_0 \cos \theta \sin \theta - 2 \sin \beta_0(\cos^2 \theta - \sin^2 \theta)) u_{\sigma\tau} \\ &= \cos(2\theta - \beta_0)(u_{\tau\tau} - u_{\sigma\sigma}) + 2 \sin(2\theta - \beta_0) u_{\sigma\tau}. \end{aligned}$$

Hence, choosing  $\theta = \beta_0/2 + \pi/4$ , the equation becomes  $u_{\sigma\tau} = 0$ , whose general solution is

$$\begin{aligned} u(s, t) &= f_1(\sigma) + f_2(\tau) = f_1(\cos \theta s + \sin \theta t) + f_2(-\sin \theta s + \cos \theta t) \\ &= f_1(\langle p, e^{i\theta} \rangle) + f_2(\langle p, ie^{i\theta} \rangle). \end{aligned}$$

■



It is easy to see that the second part of Theorem 2 is essentially a rewriting of the previous proposition.

**Remark 3.4** *The formula  $\beta = \arctan\left(\frac{u_{tt}-u_{ss}}{2u_{st}}\right)$  might be compared with the one we have in the classical (Riemannian) case:*

$$\beta = \arctan\left(\frac{\Delta u}{1 - \det \text{Hess}(u)}\right).$$

*In the classical case, the minimality is expressed by a kind of “interpolation” between the Laplace and Monge-Ampère equation. Here, we can regard the equation as an interpolation between two hyperbolic equations, the wave equation and the operator  $\partial_{st}$ .*

**Remark 3.5** *The fact that here minimal Lagrangian surfaces can be only of class of  $C^1$  makes a great contrast with the positive case, where Special Lagrangian surfaces must be analytic (the underlying equation being elliptic).*

**Example 3.6** *Taking for example  $u(s, t) = \sin s + \cos t$ , we get a doubly periodic minimal Lagrangian surface in  $\mathbb{R}^4$  or equivalently a compact minimal Lagrangian surface in  $T\mathbb{T}^2$ .*

## 4 The case of $T\mathbb{S}^2$ and normal congruences of surfaces in $\mathbb{R}^3$

It is well known that the normal congruence to a regular, oriented surface  $S$  of  $(\mathbb{R}^3, \langle \cdot, \cdot \rangle)$  defines a Lagrangian surface  $\bar{S}$  in the space  $\mathbb{L}^3$  of oriented lines of  $\mathbb{R}^3$ . The latter is naturally identified with  $T\mathbb{S}^2$  by the following

$$\mathbb{L}^3 \ni \{V + tp, t \in \mathbb{R}\} \simeq (p, V - \langle V, p \rangle p) \in T\mathbb{S}^2.$$

Since  $T\mathbb{S}^2$  is naturally and isometrically embedded in  $T\mathbb{R}^3 = \mathbb{R}^3 \times \mathbb{R}^3$  as the submanifold

$$\mathcal{S} = \{(N, Y) \in \mathbb{R}^3 \times \mathbb{R}^3, \langle N, Y \rangle = 0\},$$

we have two ways of describing a tangent vector  $\xi$  at a point  $(N, Y)$ :

- it can be seen as a couple  $\xi \simeq (\nu, \eta)$  in  $\mathbb{R}^3 \times \mathbb{R}^3$  such that  $\langle N, \nu \rangle = 0$  and  $\langle N, \eta \rangle + \langle \nu, Y \rangle = 0$ , or

- using the direct sum (1), we see that  $P\xi = \nu$  and  $K\xi = \nabla_\nu Y$  where  $s \mapsto Y(s)$  extends along the tangent direction  $\eta$ ; then  $K\xi$  is the tangential projection  $(\eta)^T = \eta - \langle \eta, N \rangle N$ .

**Lemma 4.1** *The deformation of a regular surface  $S$  of  $\mathbb{R}^3$  induces a Hamiltonian deformation of  $\tilde{S}$  in  $T\mathbb{S}^2$ .*

*Proof.* Let  $X : U \rightarrow \mathbb{R}^3$  a local parametrization of  $S$ ,  $N$  the unit normal vector field and  $h$  a compactly supported function on  $U$ ; we furthermore assume that  $X$  is a parametrization along the lines of curvatures, so that, denoting by  $\lambda$  and  $\mu$  the curvature functions, we have the two equations  $N_s = \lambda X_s$  and  $N_t = \mu X_t$ .

We consider a normal variation  $V = hN$ , where  $h$  is some smooth real map on  $U$ . Starting from  $X^\epsilon = X + \epsilon hN$ , we have

$$X_s^\epsilon = X_s + \epsilon(h_s N + h N_s) \quad X_t^\epsilon = X_t + \epsilon(h_t N + h N_t),$$

so that

$$X_s^\epsilon \times X_t^\epsilon = X_s \times X_t + \epsilon W + o(\epsilon),$$

where

$$W := h_s N \times X_t + h N_s \times X_t + h_t X_s \times N + h X_s \times N_t.$$

Consequently

$$|X_s^\epsilon \times X_t^\epsilon| = |X_s \times X_t| + \epsilon \langle N, W \rangle + o(\epsilon)$$

and

$$N^\epsilon = N + \epsilon \frac{W - \langle W, N \rangle N}{|X_s \times X_t|} + o(\epsilon) = N + \epsilon \frac{W^T}{|X_s \times X_t|} + o(\epsilon),$$

where  $W^T$  denotes again the tangential projection.

Introducing the notations  $e_1 := X_s/|X_s|$  and  $e_2 := X_t/|X_t|$ , we have

$$W^T = h_s N \times X_t + h_t X_s \times N = h_s |X_t| N \times e_2 + h_t |X_s| e_1 \times N,$$

so that

$$N^\epsilon = N - \epsilon \left( \frac{h_s}{|X_s|} e_1 + \frac{h_t}{|X_t|} e_2 \right) + o(\epsilon).$$

The next step consists of looking at the effect of this normal variation  $V = hN$  on the normal congruence. A parametrization of the normal congruence of  $X$  being  $\bar{X} = (N, X - \langle X, N \rangle N)$ , we have

$$\begin{aligned} X^\epsilon - \langle X^\epsilon, N^\epsilon \rangle N^\epsilon &= X + \epsilon hN \\ - \left\langle X + \epsilon hN, N - \epsilon \left( \frac{h_s}{|X_s|} e_1 + \frac{h_t}{|X_t|} e_2 \right) \right\rangle \left( N - \epsilon \left( \frac{h_s}{|X_s|} e_1 + \frac{h_t}{|X_t|} e_2 \right) \right) &+ o(\epsilon) \\ = X - \langle X, N \rangle N + \epsilon \left[ \left\langle X, \frac{h_s}{|X_s|} e_1 + \frac{h_t}{|X_t|} e_2 \right\rangle N + \langle X, N \rangle \left( \frac{h_s}{|X_s|} e_1 + \frac{h_t}{|X_t|} e_2 \right) \right] & \\ &+ o(\epsilon) \end{aligned}$$

so finally  $\bar{X}^\epsilon = \bar{X} + \epsilon \bar{V} + o(\epsilon)$  with

$$\begin{aligned} \bar{V} = \left( -\frac{h_s}{|X_s|} e_1 - \frac{h_t}{|X_t|} e_2, \right. \\ \left. \left\langle X, \frac{h_s}{|X_s|} e_1 + \frac{h_t}{|X_t|} e_2 \right\rangle N + \langle X, N \rangle \left( \frac{h_s}{|X_s|} e_1 + \frac{h_t}{|X_t|} e_2 \right) \right) \end{aligned}$$

where we have used the  $T\mathcal{S}$  formalism. So that

$$P\bar{V} = -\frac{h_s}{|X_s|} e_1 - \frac{h_t}{|X_t|} e_2 \text{ and } K\bar{V} = \langle X, N \rangle \left( \frac{h_s}{|X_s|} e_1 + \frac{h_t}{|X_t|} e_2 \right).$$

In order to understand the *normal* variation induced by  $\bar{V}$  on  $\bar{S}$ , we compute a basis of its normal space.

$$P\bar{X}_s = N_s = \lambda |X_s| e_1$$

$$\begin{aligned} K\bar{X}_s &= X_s - (\langle X_s, N \rangle N - \langle X, N_s \rangle N - \langle X, N \rangle N_s)^T \\ &= X_s - \langle X, N \rangle N_s = (1 - \lambda \langle X, N \rangle) |X_s| e_1 \end{aligned}$$

Analogously, we have

$$P\bar{X}_t = \mu |X_t| e_2, \quad K\bar{X}_t = (1 - \mu \langle X, N \rangle) |X_t| e_2$$

It is then obvious to compute the orthonormal basis for the normal bundle and we deduce

$$\mathbb{G}(\bar{V}, \mathbb{J}\bar{X}_s) = \Omega(\bar{V}, \bar{X}_s) = \langle X, N \rangle \lambda h_s + h_s (1 - \lambda \langle X, N \rangle) = h_s$$

and similarly  $\mathbb{G}(\bar{V}, \mathbb{J}\bar{X}_t) = h_t$ . This means that  $\bar{V}^\perp = \mathbb{J}Dh$ , i.e. the vector field  $\bar{V}^\perp$  is Hamiltonian. ■

**Lemma 4.2** *Let  $S$  be a surface in  $\mathbb{R}^3$  and  $\bar{S}$  its normal congruence. We denote by  $\mathcal{A}(\bar{S})$  the area with respect to the metric  $\mathbb{G}$  and by  $\mathcal{F}(S)$  the functional defined by  $\mathcal{F}(S) := \int_S \sqrt{H^2 - K} dA$ , where  $H$  and  $K$  are respectively the mean curvature and the Gauss curvature of  $S$ . Then*

$$\mathcal{A}(\bar{S}) = \mathcal{F}(S).$$

*Proof.* From the expressions for  $\bar{X}_s$  and  $\bar{X}_t$  computed in the proof of Lemma 4.1, we obtain the coefficients of the first fundamental form of the immersion  $\bar{X}$ :

$$\bar{E} = \bar{G} = 0, \quad \bar{F} = \Omega(\mathbb{J}\bar{X}_s, \bar{X}_t) = (\mu - \lambda)|X_s||X_t|.$$

It follows that

$$\begin{aligned} \int_U \sqrt{|\bar{E}\bar{G} - \bar{F}^2|} ds dt &= \int_U |\bar{F}| ds dt = \int_U |\lambda - \mu| \sqrt{EG - F^2} ds dt \\ &= \int_{X(U)} \sqrt{H^2 - K} dA \end{aligned}$$

so  $\mathcal{A}(\bar{S}) = \mathcal{F}(S)$ . ■

Lemmas 4.1 and 4.2 prove that the normal congruence of a surface  $S$  of  $\mathbb{R}^3$  is Hamiltonian stationary if and only if  $S$  is a critical point of  $\mathcal{F}$ . On the other hand, we know by Proposition 2.1 that rank one Lagrangian surfaces of  $T\mathbb{S}^2$  are Hamiltonian stationary. The next lemma provides a geometric interpretation of the rank one condition:

**Lemma 4.3** *A non-planar surface  $S$  of  $\mathbb{R}^3$  is developable if and only if its normal congruence defines a rank one Lagrangian surface in  $T\mathbb{S}^2$ , i.e. is the normal bundle of some curve of  $\mathbb{S}^2$ .*

*Proof.* By definition, a developable surface has vanishing Gauss curvature, which implies that the Gauss image is a curve (or a single point) in  $\mathbb{S}^2$ . As the Gauss map of  $S$  is nothing but the projection of  $\bar{S}$  on the base  $\mathbb{S}^2$ , the result follows.

Finally, putting all these facts together we get:

**Corollary 4.4** *A developable surface of  $S$  of  $\mathbb{R}^3$  is a critical point of the functional  $\mathcal{F}$ .*

Finally everything in Section 4 can be readily adapted to the case of  $T\mathbb{H}^2$ , which is identified with the set of positive time lines  $\mathbb{L}_+^3$  of the Minkowski space  $(\mathbb{R}^{2,1}, \langle \cdot, \cdot \rangle_1)$ , by the following

$$\mathbb{L}_+^3 \ni \{V + tp, t \in \mathbb{R}\} \simeq (p, V - \langle V, p \rangle_1 p) \in T\mathbb{H}^2.$$

Here,  $\mathbb{H}^2$  denotes the hyperboloid model of the hyperbolic plane, that is the space-like quadric

$$\mathbb{H}^2 := \{p \in \mathbb{R}^{2,1}, \langle p, p \rangle_1 = -1, p_3 > 0\}.$$

We leave to the Reader the easy task to check that a developable space-like surface of  $(\mathbb{R}^{2,1}, \langle \cdot, \cdot \rangle_1)$  is a critical point of the functional equivalent to  $\mathcal{F}$  in  $\mathbb{R}^{2,1}$ .

## References

- [Ch] B.-Y. Chen, *Riemannian geometry of Lagrangian submanifolds*, Taiwanese J. Math **5** (2001) 681–723
- [Do] P. Dombrowski, *On the geometry of the tangent bundle*, J. Reine Angew. Math. **210** (1962) 73–88
- [Dg] Y. Dong, *On indefinite special Lagrangian submanifolds in indefinite complex euclidean spaces*, [arXiv:0805.2718](https://arxiv.org/abs/0805.2718).
- [GK1] B. Guilfoyle, W. Klingenberg, *An indefinite Kähler metric on the space of oriented lines*, J. London Math. Soc. **72** (2005) 497–509.
- [GK2] B. Guilfoyle, W. Klingenberg, *On area-stationary surfaces in certain neutral Kähler 4-manifolds*, Beiträge Algebra Geom. **49**, No. 2 (2008) 481–490.
- [HL] R. Harvey, H. B. Lawson, *Calibrated geometries*, Acta Math. **148** (1982) 47–157.
- [La] J. Lafontaine, *Some relevant Riemannian geometry*, in *Holomorphic curves in symplectic geometry*, J. Lafontaine and M. Audin ed. Birkhäuser

- [Ko] O. Kowalski, *Curvature of the induced Riemannian metric on the tangent bundle of a Riemannian manifold* J. Reine Angew. Math. **250** (1971) 124–129
- [S] R. Schoen, *Special Lagrangian submanifolds*, Global theory of minimal surfaces, 655–666, Clay Math. Proc., **2**, Amer. Math. Soc., Providence, RI, 2005
- [SW] R. Schoen, J. Wolfson *Minimizing volume among Lagrangian submanifolds*. Proc. Symp. Pure Math. **65** Amer. Math. Soc., Providence, RI (1999)

Henri Anciaux  
 Department of Mathematics and Computing  
 Institute of Technology, Tralee  
 Co. Kerry, Ireland  
 henri.anciaux@ittralee.ie

Brendan Guilfoyle  
 Department of Mathematics and Computing  
 Institute of Technology, Tralee  
 Co. Kerry, Ireland  
 brendan.guilfoyle@ittralee.ie

Pascal Romon  
 Université de Paris-Est Marne-la-Vallée  
 5, bd Descartes, Champs-sur-Marne  
 77454 Marne-la-Vallée cedex 2, France  
 pascal.romon@univ-mlv.fr